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Some Modficiations in Method of Improving the Orbits of Artificial Earth Satellites

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A method involving the use of a rotating system of coordinates is proposed for improving the orbital elements of artificial satellites.

The problem of orbit improvement amounts to the determination of the system of elements that best fits the observations. The orbits of artificial satellites can be improved by using any of the classical methods of orbit correction developed for the study of the motion of celestial bodies.

Let us use the method set forth in M. F. Subbotin's *Course in Celestial Mechanics* (1941) under the designation of "second method." The basic formulas that determine the position of a satellite (α,δ) on the celestial sphere are

$$\begin{aligned} x - X &= \rho \cos\delta \cos\alpha \\ y - Y &= \rho \cos\delta \sin\alpha \\ z - Z &= \rho \sin\delta \\ \rho^2 &= (x - X)^2 + (y - Y)^2 + (z - Z)^2 \end{aligned} \tag{1}$$

where ρ is the distance from the observer to the satellite, x,y,z are the rectangular geocentric coordinates of the satellite, and X,Y,Z are the coordinates of the observing station. By differentiating and transforming formulas (1), we obtain the equations of condition connecting the rectangular-coordinate corrections with the deviations of the calculated positions of the body (α_c,δ_c) from the observed positions (α_0,δ_0):

$$\begin{aligned} \rho \cos\delta d\alpha &= -\sin\alpha dx + \cos\alpha dy \\ \rho d\delta &= -\sin\delta \cos\alpha dx - \sin\delta \sin\alpha dy + \cos\delta dz \end{aligned} \tag{2}$$

where $dx,dy,$ and dz are expressed in terms of the partial derivatives of the rectangular coordinates x,y,z with respect to the orbital elements, as follows:

$$\begin{aligned} dx &= \frac{\partial x}{\partial \Omega} d\Omega + \frac{\partial x}{\partial i} di + \frac{dx}{\partial M_0} dM_0 + \frac{\partial x}{d\omega} d\omega + \frac{\partial x}{\partial \varphi} d\varphi + \frac{\partial x}{\partial n} dn \\ dy &= \frac{\partial y}{\partial \Omega} d\Omega + \frac{\partial y}{\partial i} di + \frac{\partial y}{\partial M_0} dM_0 + \frac{\partial y}{\partial \omega} d\omega + \frac{\partial y}{\partial \varphi} d\varphi + \frac{\partial y}{\partial n} dn \\ dz &= \frac{\partial z}{\partial \Omega} d\Omega + \frac{\partial z}{\partial i} di + \frac{\partial z}{\partial M_0} dM_0 + \frac{\partial z}{\partial \omega} d\omega + \frac{\partial z}{\partial \varphi} d\varphi + \frac{\partial z}{\partial n} dn \end{aligned} \tag{3}$$

where Ω is the longitude of the ascending node from the point of vernal equinox, i the orbit's inclination to the equator, M_0 the mean anomaly at time t_0, ω the argument of perigee, φ the angle of eccentricity (related to the eccentricity by the formula $e = \sin\varphi$), and n the diurnal motion of the satellite. The correction of the orbital elements reduces to the calculation of these partial derivatives and to the determination of the corrections of the elements $d\Omega$, di, etc., by the method of least squares.

The special feature of the "second method" consists in the fact that the rectangular coordinates of the celestial body x,y,z, as well as the corresponding formulas for $\dot{x},\dot{y},\dot{z}$, are expressed in terms of the *true* anomaly by the formulas:

$$\begin{aligned} x &= r(\cos u \cos\Omega - \sin u \sin\Omega \cos i) \\ y &= r(\cos u \sin\Omega + \sin u \cos\Omega \cos i) \\ z &= r \sin u \cos i \\ u &= v + \omega \\ r &= \frac{\bar{a} \cos^2\varphi}{1 + \sin\varphi \cos v} \end{aligned} \tag{4}$$

or

$$\begin{aligned} x &= r \sin a \sin(A + u) \\ y &= r \sin b \sin(B + u) \\ z &= r \sin c \sin(C + u) \end{aligned} \tag{5}$$

$$\begin{aligned} \dot{x} &= n\bar{a}[\sin a \cos(A + u)\sec\varphi + \sin a \cos(A + \omega) \tan\varphi] \\ \dot{y} &= n\bar{a}[\sin b \cos(B + u)\sec\varphi + \sin b \cos(B + \omega) \tan\varphi] \\ \dot{z} &= n\bar{a}[\sin c \cos(C + u)\sec\varphi + \sin c \cos(C + \omega) \tan\varphi] \end{aligned} \tag{6}$$

where $\bar{a}$ is the semimajor axis of the orbit.

Using formulas (5) and (6) we obtain the expression for the partial derivatives in terms of the true anomaly. These formulas can be found in M. F. Subbotin's *Course in Celestial Mechanics*.[4]

However, it is rather inconvenient to use these formulas in practice owing to their unwieldiness. The formulas

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can be greatly simplified by a particular choice of coordinate axes.

Let us direct the OX axis at the ascending node of the orbit Ω, and not at the point of vernal equinox, as customary in classical methods. The direction of the other axes is left unchanged.

Then formulas (1) and (4) assume the form

$$\begin{aligned} x - X &= \rho \cos\delta \cos(\alpha - \Omega) \\ y - Y &= \rho \cos\delta \sin(\alpha - \Omega) \\ z - Z &= \rho \sin\delta \\ \rho^2 &= (x - X)^2 + (y - Y)^2 + (z - Z)^2 \end{aligned} \tag{7}$$

$$\begin{aligned} x &= r \cos u \\ y &= r \cos i \sin u \\ z &= r \sin i \sin u \\ r &= \frac{\bar{a} \cos^2\varphi}{1 + \sin\varphi \cos v} \end{aligned} \tag{8}$$

The coordinates X,Y,Z of the observing station will be determined in the new system by the formulas

$$\begin{aligned} X &= R \cos\varphi' \cos(s - \Omega) \\ Y &= R \cos\varphi' \sin(s - \Omega) \\ Z &= R \sin\varphi' \\ R^2 &= X^2 + Y^2 + Z^2 \end{aligned} \tag{9}$$

where φ' is the geocentric latitude of the point of observation and s the local sidereal time.

Let us note that in this system the coordinates x,y,z of the satellite do not depend on node longitude. However, this element enters the coordinates X,Y,Z of the observing station. Therefore, not x,y,z, but $x-X$, $y-Y$, and $z-Z$ have to be differentiated with respect to the orbital elements. In this case, formulas (3) assume the form

$$\begin{aligned} d(x - X) &= \frac{\partial X}{d\Omega} d\Omega + \frac{\partial x}{\partial i} di + \frac{\partial x}{\partial M_0} dM_0 + \frac{\partial x}{\partial \omega} d\omega + \frac{\partial x}{\partial \varphi} d\varphi + \frac{\partial x}{\partial n} dn \\ d(y - Y) &= \frac{\partial Y}{\partial\Omega} d\Omega + \frac{\partial y}{\partial i} di + \frac{\partial y}{\partial M_0} dM_0 + \frac{\partial y}{\partial \omega} d\omega + \frac{\partial y}{\partial \varphi} d\varphi + \frac{\partial y}{\partial n} dn \\ d(z - Z) &= \frac{\partial Z}{\partial\Omega} d\Omega + \frac{\partial z}{\partial i} di + \frac{\partial z}{\partial M_0} dM_0 + \frac{\partial z}{\partial \omega} d\omega + \frac{\partial z}{\partial \varphi} d\varphi + \frac{\partial z}{\partial n} dn \end{aligned} \tag{10}$$

The formulas for the differential coefficients of $di,dM_0,d\omega$, $d\varphi$, and dn can be obtained from the corresponding coefficients of the second method by setting $\Omega = 0$, $a = \pi/2$, $b = (\pi/2) + i$, $c = i$, $A = \pi/2$, $B = 0$, $C = 0$.

Here are the formulas for the coefficients of all the elements:

$$\begin{aligned} &\frac{\partial X}{\partial\Omega} = -Y \quad \frac{\partial Y}{\partial\Omega} = X \quad \frac{\partial Z}{\partial\Omega} = 0 \\ &\frac{\partial x}{\partial i} = 0 \quad \frac{\partial y}{\partial i} = -z \quad \frac{\partial z}{\partial i} = y \\ &\frac{\partial x}{\partial M_0} = \frac{\dot{x}}{n} = -\frac{\bar{a}}{\cos\varphi}[\sin u + \sin\varphi \sin\omega] \\ &\frac{\partial y}{\partial M_0} = \frac{\dot{y}}{n} = \frac{\bar{a}}{\cos\varphi}[\cos u + \sin\varphi \cos\omega] \\ &\frac{\partial z}{\partial M_0} = \frac{\dot{z}}{n} = \frac{\bar{a}}{\cos\varphi}[\cos u + \sin\varphi \cos\omega] \\ &\partial x/\partial\omega = -r \sin u \quad \partial y/\partial\omega = r \cos i \cos u \\ &\partial z/\partial\omega = r \sin i \cos u \\ &\partial x/\partial\varphi = -\bar{a}[\sin u \sin E + \cos\varphi \cos\omega] \\ &\partial y/\partial\varphi = \bar{a} \cos i[\cos u \sin E - \cos\varphi \sin\omega] \\ &\partial z/\partial\varphi = \bar{a} \sin i[\cos u \sin E - \cos\varphi \sin\omega] \\ &\partial x/\partial n = \frac{\dot{x}}{n}(t - t_0) - \frac{2}{3n}x, \ \frac{\partial y}{\partial n} = \frac{\dot{y}}{n}(t - t_0) - \frac{2}{3n}y, \ \frac{\partial z}{\partial n} = \frac{\dot{z}}{n}(t - t_0) - \frac{2}{3n}z \end{aligned} \tag{11}$$

where t is the moment of observation of the satellite.

The problem is completely solved by substituting the formulas for the coefficients (11) in the equations of condition, which are written in the new system of coordinates in the form

$$\begin{aligned} \rho \cos\delta d(\alpha - \Omega) &= -\sin(\alpha - \Omega)dx + \cos(\alpha - \Omega)dy + \rho \cos\delta d\Omega \\ \rho d\delta &= -\sin\delta \cos(\alpha - \Omega)dx - \sin\delta \sin(\alpha - \Omega)dy + \cos\delta dz \end{aligned} \tag{12}$$

Formulas (10), (11), and (12) were listed without derivation in one of our earlier papers (Batrakov and Sochilina).[1]

The obtained formulas can be somewhat modified by using the identities

$$\begin{aligned} X \cos(\alpha - \Omega) + Y \sin(\alpha - \Omega) + \rho \cos\delta &= x \cos(\alpha - \Omega) + y \sin(\alpha - \Omega) \\ Y \sin\delta \cos(\alpha - \Omega) - X \sin\delta \sin(\alpha - \Omega) &= y \sin\delta \cos(\alpha - \Omega) - x \sin\delta \sin(\alpha - \Omega) \end{aligned} \tag{13}$$

Then, the differential coefficients X and Y of $d\Omega$ in formulas (11) can be replaced by x and y, respectively, and formulas (12) assume the form

$$\begin{aligned} \rho \cos\delta d\alpha &= -\sin(\alpha - \Omega)dx + \cos(\alpha - \Omega)dy \\ \rho d\delta &= -\sin\delta \cos(\alpha - \Omega)dx - \sin\delta \sin(\alpha - \Omega)dy + \cos\delta dz \end{aligned} \tag{14}$$

The elements of a satellite vary quite rapidly due to the earth's oblateness and atmospheric drag. Therefore, the chosen system of coordinates is a moving one; it rotates together with the node of the satellite's orbit.

In computing the coefficients of the equations of condition, the instantaneous values of the satellite elements are taken, i.e., the values at the moment of observation. The perturbations of the elements are calculated by analytic formulas, obtained by various authors: Proskurin and Batrakov,[3] Zhongolovich,[2] Kozai,[5] and others.

The effect of atmospheric drag on satellite motion can be accounted for most simply by introducing empirical terms. Since the drag has maximal effect on the diurnal motion of the satellite, we shall give the formula for the differential coefficients of these additional terms only. Suppose n is represented in the form

$$n = n_0 + 2n_0'(t - t_0) + 3n_0''(t - t_0)^2 \tag{15}$$

Then the coefficients of the corrections dn' and dn'' are written in the form:

$$\begin{aligned} \frac{\partial x}{\partial n'} &= \left[\frac{\dot{x}}{n}(t - t_0) - \frac{4}{3n}x\right](t - t_0) \\ \frac{\partial y}{\partial n'} &= \left[\frac{\dot{y}}{n}(t - t_0) - \frac{4}{3n}y\right](t - t_0) \\ \frac{\partial z}{\partial n'} &= \left[\frac{\dot{z}}{n}(t - t_0) - \frac{4}{3n}z\right](t - t_0) \\ \frac{\partial x}{\partial n''} &= \left[\frac{\dot{x}}{n}(t - t_0) - \frac{2}{n}x\right](t - t_0)^2 \\ \frac{\partial y}{\partial n''} &= \left[\frac{\dot{y}}{n}(t - t_0) - \frac{2}{n}y\right](t - t_0)^2 \\ \frac{\partial z}{\partial n''} &= \left[\frac{\dot{z}}{n}(t - t_0) - \frac{2}{n}z\right](t - t_0)^2 \end{aligned} \tag{16}$$

The formulas for the empirical terms of other elements can be obtained analogously.

In conclusion we would like to express deep gratitude to Yu. V. Batrakov for valuable advice relating to the present work.

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Decay of Shock Waves in Stationary Flows

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THE asymptotic laws of extinction of nonstationary shock waves at great distances frôm the point of explosion were found by Landau,[1] Khristianovich,[2] Sedov,[3] and Whitham[4] on the assumption that the medium is homogeneous (uniform). A study of the propagation of shock waves in a nonuniform moving medium in the realm of geometrical acoustics was carried out in the work of Keller.[5] Articles by Gubkin,[6] Otterman,[7] Polianskii,[8] and the author[9] are devoted to a refinement of the acoustical theory in which the nonlinear nature of the gasdynamic equations is not taken into account.

The laws of extinction of weak shock waves in steady-state supersonic flows have been studied less. They were established by Landau[1] and Whitham[10] for plane-parallel and axially symmetric uniform flows. The behavior of plane shock waves was studied in the work of Friedrichs[11] with a higher degree of accuracy in second approximation.

The present work explains the main features of the development of shock waves of small amplitude in nonuniform steady-state supersonic flows; the width of the perturbed region of flow is assumed small as compared with the radius of curvature of the density jump and with the distance at which the parameters of the initial medium change essentially. On the basis of the investigation, it was proposed that each small element of the perturbed region in first approximation may be examined as a nonstationary Rieman wave carried laterally by the uniform flow. The laws of the variation of the parameters of the medium at the shock fronts may be then used with the method similar to that used by Landau.[1] It appears, however, that the results obtained in Ref. 9 for nonstationary shock waves moving in a nonuniform medium cannot be applied directly for a computation of steady-state supersonic flows. The cause for this is included in the different laws determining the variation of the width of the perturbed region in steady-state and nonstationary processes.

1. The initial system of gasdynamic equations may be written

$$v_j \frac{\partial v_i}{\partial x_j} + \frac{1}{\rho}\frac{\partial p}{\partial x_i} = g_i \qquad \frac{\partial \rho v_j}{\partial x_j} = 0$$
$$v_j(\partial s/\partial x_j) = 0 \qquad p = p(\rho, s) \tag{1.1}$$

Here v_i, g_i, p, ρ, and s designate the velocity components of the flow and of the mass forces, pressure, density, and entropy at the point with the rectangular coordinates x_i, respectively. The usual tensor notation of the sums is used according to the repeating subscripts i, j, which take on the values 1, 2, 3.

Subsequently, we shall examine only supersonic flows. The required system of equations (1.1) will then be of the hyperbolic type. The equation which determines the C_+-characteristic surfaces $\varphi(x_i) = 0$ of this system, may therefore be written

$$v_j n_j + a = 0 \qquad a = \sqrt{(\partial p/\partial \rho)_s} \tag{1.2}$$

Here a is the speed of sound and n_i the components of the normal to these surfaces, for which the following formula holds:

$$n_i = \frac{\partial \varphi/\partial x_i}{\sqrt{(\partial \varphi/\partial x_j)^2}}$$

The system of gasdynamic equations (1.1) on the C_+-characteristics takes on the form

$$(v_i + a n_i)\frac{\partial p}{\partial x_i} + a\rho(a\delta_{ij} + n_i v_j)\frac{\partial v_i}{\partial x_j} = \rho a n_i g_i$$
$$\delta_{ij} = 1 \text{ when } i = j \qquad \delta_{ij} = 0 \text{ when } i \neq j \tag{1.3}$$

Equation (1.3) contains the derivatives of the sought functions only along the C_+-characteristic surfaces.

We proceed to a study of the behavior of weak shock waves in a nonuniform flow. We consider that, in an unperturbed state, the pressure p_0, the density ρ_0, the speed of sound a_0, and the velocity components of the flow v_{0i} are given as functions of the coordinates x_i. Because of the smallness of the amplitude of the density jump, the relative variations of all the gas parameters at its front are small. Therefore, we assume

$$p = p_0 + p' \qquad \rho = \rho_0 + \rho'$$
$$a = a_0 + a' \qquad v_i = v_{0i} + v_i'$$

Here p', ρ', a', and v_i' are the excess pressure, density, speed of sound, and velocity components of the particles in the region of perturbed flow. The dimensionless values

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